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Baccalauréat International
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MARKSCHEME

November 2014

MATHEMATICS

Higher Level

Paper 1

20 pages

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**; often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown.

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Mark according to RM™ Assessor instructions and the document "**Mathematics HL: Guidance for e-marking November 2014**". It is essential that you read this document before you start marking. In particular, please note the following:

- Marks must be recorded using the annotation stamps. Please check that you are entering marks for the right question.
- If a part is **completely correct**, (and gains all the "must be seen" marks), use the ticks with numbers to stamp full marks.
- If a part is completely wrong, stamp ***A0*** by the final answer.
- If a part gains anything else, it **must** be recorded using **all** the annotations.
- All the marks will be added and recorded by RM™ Assessor.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *eg* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*eg* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working.*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.

4 Implied marks

Implied marks appear in ***brackets eg (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). A candidate should be penalized only once for a particular mis-read. Use the* ***MR*** *stamp to indicate that this has been a misread. Then deduct the first of the marks to be awarded, even if this is an* ***M*** *mark, but award all others so that the candidate only loses one mark.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*eg* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. In such cases the annotation DM should be used and a brief ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

*Unless the question specifies otherwise, **accept** equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

$$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(=10\cos(5x-3)\big) \qquad \textbf{\textit{A1}}$$

Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

*Candidates should **NO LONGER** be penalized for an accuracy error **(AP)**.*

*If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy. When this is not specified in the question, all numerical answers should be given exactly or correct to three significant figures. Please check work carefully for **FT**.*

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Calculators

No calculator is allowed. The use of any calculator on paper 1 is malpractice, and will result in no grade awarded. If you see work that suggests a candidate has used any calculator, please follow the procedures for malpractice. Examples: finding an angle, given a trig ratio of 0.4235.

13 More than one solution

Where a candidate offers two or more different answers to the same question, an examiner should only mark the first response unless the candidate indicates otherwise.

14. Candidate work

Candidates are meant to write their answers to Section A on the question paper (QP), and Section B on answer booklets. Sometimes, they need more room for Section A, and use the booklet (and often comment to this effect on the QP), or write outside the box. This work should be marked.

The instructions tell candidates not to write on Section B of the QP. Thus they may well have done some rough work here which they assume will be ignored. If they have solutions on the answer booklets, there is no need to look at the QP. However, if there are whole questions or whole part solutions missing on answer booklets, please check to make sure that they are not on the QP, and if they are, mark those whole questions or whole part solutions that have not been written on answer booklets.

SECTION A

1. (a) $g(x) = \frac{1}{x+3} + 1$ ***A1A1***

> **Note:** Award ***A1*** for $x+3$ in the denominator and ***A1*** for the "+1".

[2 marks]

(b) $x = -3$ ***A1***
$y = 1$ ***A1***

[2 marks]

Total [4 marks]

2. (a) using the formulae for the sum and product of roots:

(i) $\alpha + \beta = 4$ ***A1***

(ii) $\alpha\beta = \frac{1}{2}$ ***A1***

> **Note:** Award ***A0A0*** if the above results are obtained by solving the original equation (except for the purpose of checking).

[2 marks]

(b) **METHOD 1**

required quadratic is of the form $x^2 - \left(\frac{2}{\alpha} + \frac{2}{\beta}\right)x + \left(\frac{2}{\alpha}\right)\left(\frac{2}{\beta}\right)$ ***(M1)***

$q = \frac{4}{\alpha\beta}$

$q = 8$ ***A1***

$p = -\left(\frac{2}{\alpha} + \frac{2}{\beta}\right)$

$= -\frac{2(\alpha+\beta)}{\alpha\beta}$ ***M1***

$= -\frac{2 \times 4}{\frac{1}{2}}$

$p = -16$ ***A1***

> **Note:** Accept the use of exact roots

continued …

Question 2 continued

METHOD 2

replacing x with $\frac{2}{x}$ ***M1***

$2\left(\frac{2}{x}\right)^2 - 8\left(\frac{2}{x}\right) + 1 = 0$

$\frac{8}{x^2} - \frac{16}{x} + 1 = 0$ ***(A1)***

$x^2 - 16x + 8 = 0$

$p = -16$ and $q = 8$ ***A1A1***

Note:	Award ***A1A0*** for $x^2 - 16x + 8 = 0$ *ie*, if $p = -16$ and $q = 8$ are not explicitly stated.

[4 marks]

Total [6 marks]

3. METHOD 1

$\left|\overrightarrow{OP}\right| = \sqrt{(1+s)^2+(3+2s)^2+(1-s)^2} \quad \left(= \sqrt{6s^2+12s+11}\right)$ ***A1***

Note: Award ***A1*** if the square of the distance is found.

EITHER

attempt to differentiate: $\frac{\mathrm{d}}{\mathrm{d}s}\left|\overrightarrow{OP}\right|^2 \; (= 12s + 12)$ ***M1***

attempting to solve $\frac{\mathrm{d}}{\mathrm{d}s}\left|\overrightarrow{OP}\right|^2 = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt to differentiate: $\frac{\mathrm{d}}{\mathrm{d}s}\left|\overrightarrow{OP}\right| \quad \left(= \frac{6s + 6}{\sqrt{6s^2 + 12s + 11}}\right)$ ***M1***

attempting to solve $\frac{\mathrm{d}}{\mathrm{d}s}\left|\overrightarrow{OP}\right| = 0$ for s ***(M1)***

$s = -1$ ***(A1)***

OR

attempt at completing the square: $\left(\left|\overrightarrow{OP}\right|^2 = 6(s+1)^2 + 5\right)$ ***M1***

minimum value ***(M1)***
occurs at $s = -1$ ***(A1)***

THEN

the minimum length of $\overrightarrow{OP}$ is $\sqrt{5}$ ***A1***

METHOD 2

the length of $\overrightarrow{OP}$ is a minimum when $\overrightarrow{OP}$ is perpendicular to $\begin{pmatrix}1\\2\\-1\end{pmatrix}$ ***(R1)***

$\begin{pmatrix}1+s\\3+2s\\1-s\end{pmatrix} \bullet \begin{pmatrix}1\\2\\-1\end{pmatrix} = 0$ ***A1***

attempting to solve $1+s+6+4s-1+s=0 \; (6s+6=0)$ for s ***(M1)***
$s = -1$ ***(A1)***
$\left|\overrightarrow{OP}\right| = \sqrt{5}$ ***A1***

Total [5 marks]

4. (a) (i) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5$

$= 0.7$ ***A1***

(ii) use of $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A)\mathrm{P}(B)$ ***(M1)***

$\mathrm{P}(A \cup B) = 0.2 + 0.5 - 0.1$

$= 0.6$ ***A1***

[4 marks]

(b) $\mathrm{P}(A \mid B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$

$\mathrm{P}(A \mid B)$ is a maximum when $\mathrm{P}(A \cap B) = \mathrm{P}(A)$

$\mathrm{P}(A \mid B)$ is a minimum when $\mathrm{P}(A \cap B) = 0$

$0 \le \mathrm{P}(A \mid B) \le 0.4$ ***A1A1A1***

Note: ***A1*** for each endpoint and ***A1*** for the correct inequalities.

[3 marks]

Total [7 marks]

5. use of the quotient rule or the product rule ***M1***

$$C'(t) = \frac{(3+t^2) \times 2 - 2t \times 2t}{(3+t^2)^2} \left(= \frac{6-2t^2}{(3+t^2)^2} \right) \text{ or } \frac{2}{3+t^2} - \frac{4t^2}{(3+t^2)^2}$$ ***A1A1***

Note: Award ***A1*** for a correct numerator and ***A1*** for a correct denominator in the quotient rule, and ***A1*** for each correct term in the product rule.

attempting to solve $C'(t) = 0$ for t ***(M1)***

$t = \pm\sqrt{3}$ (minutes) ***A1***

$C(\sqrt{3}) = \dfrac{\sqrt{3}}{3}$ $(\mathrm{mg\,l^{-1}})$ or equivalent. ***A1***

Total [6 marks]

6. $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}}$ ***A1***

$\mathrm{d}x = 2(u-1)\,\mathrm{d}u$

Note: Award the ***A1*** for any correct relationship between dx and du.

$\int \frac{\sqrt{x}}{1+\sqrt{x}}\,\mathrm{d}x = 2\int \frac{(u-1)^2}{u}\,\mathrm{d}u$ ***(M1)A1***

Note: Award the ***M1*** for an attempt at substitution resulting in an integral only involving u.

$= 2\int u - 2 + \frac{1}{u}\,\mathrm{d}u$ ***(A1)***

$= u^2 - 4u + 2\ln u\,(+C)$ ***A1***

$= x - 2\sqrt{x} - 3 + 2\ln\left(1+\sqrt{x}\right)(+C)$ ***A1***

Note: Award the ***A1*** for a correct expression in x, but not necessarily fully expanded/simplified.

Total [6 marks]

7. (a) $p'(3) = f'(3)\,g(3) + g'(3)\,f(3)$ ***(M1)***

Note: Award ***M1*** if the derivative is in terms of x or 3.

$= 2\times 4 + 3\times 1$

$= 11$ ***A1***

[2 marks]

(b) $h'(x) = g'\left(f(x)\right)f'(x)$ ***(M1)(A1)***

$h'(2) = g'(1)\,f'(2)$ ***A1***

$= 4\times 4$

$= 16$ ***A1***

[4 marks]

Total [6 marks]

8. let $\mathrm{P}(n)$ be the proposition that $(2n)! \geq 2^n (n!)^2$, $n \in \mathbb{Z}^+$

consider $\mathrm{P}(1)$:

$2! = 2$ and $2^1 \left(1!\right)^2 = 2$ so $\mathrm{P}(1)$ is true ***R1***

assume $\mathrm{P}(k)$ is true *ie* $(2k)! \geq 2^k (k!)^2$, $k \in \mathbb{Z}^+$ ***M1***

Note: Do not award ***M1*** for statements such as "let $n = k$".

consider $\mathrm{P}(k+1)$:

$\left(2(k+1)\right)! = (2k+2)(2k+1)(2k)!$ ***M1***

$\left(2(k+1)\right)! \geq (2k+2)(2k+1)(k!)^2 2^k$ ***A1***

$= 2(k+1)(2k+1)(k!)^2 2^k$

$> 2^{k+1}(k+1)(k+1)(k!)^2$ since $2k+1 > k+1$ ***R1***

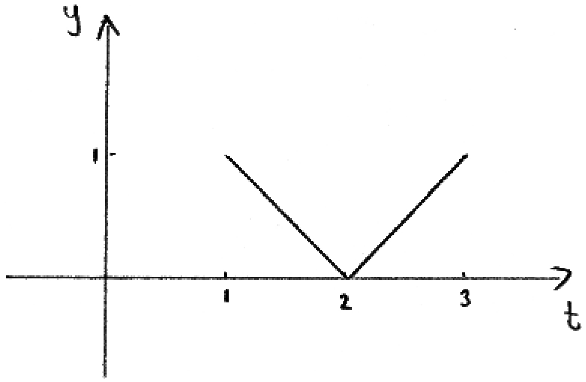
$= 2^{k+1}\left((k+1)!\right)^2$ ***A1***

$\mathrm{P}(k+1)$ is true whenever $\mathrm{P}(k)$ is true and $\mathrm{P}(1)$ is true, so $\mathrm{P}(n)$ is true for $n \in \mathbb{Z}^+$ ***R1***

Note: To obtain the final ***R1***, four of the previous marks must have been awarded.

Total [7 marks]

9. (a)



$|2-t|$ correct for $[1, 2]$ ***A1***

$|2-t|$ correct for $[2, 3]$ ***A1***

[2 marks]

(b) **EITHER**

let q_1 be the lower quartile and let q_3 be the upper quartile

let $d = 2 - q_1 \left(= q_3 - 2\right)$ and so $\text{IQR} = 2d$ by symmetry

use of area formulae to obtain $\frac{1}{2}d^2 = \frac{1}{4}$

(or equivalent) ***M1A1***

$d = \frac{1}{\sqrt{2}}$ or the value of at least one q. ***A1***

OR

let q_1 be the lower quartile

consider $\int_1^{q_1} (2-t)\,dt = \frac{1}{4}$ ***M1A1***

obtain $q_1 = 2 - \frac{1}{\sqrt{2}}$ ***A1***

THEN

$\text{IQR} = \sqrt{2}$ ***A1***

Note: Only accept this final answer for the ***A1***.

[4 marks]

Total [6 marks]

10. (a) use of the addition principle with 3 terms *(M1)*

to obtain ${}^{4}C_3 + {}^{5}C_3 + {}^{6}C_3$ $(= 4 + 10 + 20)$ *A1*

number of possible selections is 34 *A1*

[3 marks]

(b) **EITHER**

recognition of three cases: (2 odd and 2 even or 1 odd and 3 even or 0 odd and 4 even) *(M1)*

$\left({}^{5}C_2 \times {}^{4}C_2\right) + \left({}^{5}C_1 \times {}^{4}C_3\right) + \left({}^{5}C_0 \times {}^{4}C_4\right)$ $(= 60 + 20 + 1)$ *(M1)A1*

OR

recognition to subtract the sum of 4 odd and 3 odd and 1 even from the total *(M1)*

${}^{9}C_4 - {}^{5}C_4 - \left({}^{5}C_3 \times {}^{4}C_1\right)$ $(= 126 - 5 - 40)$ *(M1)A1*

THEN

number of possible selections is 81 *A1*

[4 marks]

Total [7 marks]

SECTION B

11. (a) (i) $x = \mathrm{e}^{3y+1}$ ***M1***

Note: The ***M1*** is for switching variables and can be awarded at any stage. Further marks do not rely on this mark being awarded.

taking the natural logarithm of both sides and attempting to transpose ***M1***

$\left(f^{-1}(x)\right) = \frac{1}{3}(\ln x - 1)$ ***A1***

(ii) $x \in \mathbb{R}^+$ or equivalent, for example $x > 0$. ***A1***

[4 marks]

(b) $\ln x = \frac{1}{3}(\ln x - 1) \Rightarrow \ln x - \frac{1}{3}\ln x = -\frac{1}{3}$ (or equivalent) ***M1A1***

$\ln x = -\frac{1}{2}$ (or equivalent) ***A1***

$x = \mathrm{e}^{-\frac{1}{2}}$ ***A1***

coordinates of P are $\left(\mathrm{e}^{-\frac{1}{2}}, -\frac{1}{2}\right)$ ***A1***

[5 marks]

(c) coordinates of Q are $(1, 0)$ seen anywhere ***A1***

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$ ***M1***

at Q, $\frac{\mathrm{d}y}{\mathrm{d}x} = 1$ ***A1***

$y = x - 1$ ***AG***

[3 marks]

continued ...

Question 11 continued

(d) let the required area be A

$A = \int_1^e x - 1\mathrm{d}x - \int_1^e \ln x\mathrm{d}x$ ***M1***

Note: The ***M1*** is for a difference of integrals. Condone absence of limits here.

attempting to use integration by parts to find $\int \ln x\mathrm{d}x$ ***(M1)***

$= \left[\frac{x^2}{2} - x\right]_1^{\mathrm{e}} - [x\ln x - x]_1^{\mathrm{e}}$ ***A1A1***

Note: Award ***A1*** for $\frac{x^2}{2} - x$ and ***A1*** for $x\ln x - x$.

Note: The second ***M1*** and second ***A1*** are independent of the first ***M1*** and the first ***A1***.

$= \frac{\mathrm{e}^2}{2} - \mathrm{e} - \frac{1}{2}\left(= \frac{\mathrm{e}^2 - 2\mathrm{e} - 1}{2}\right)$ ***A1***

[5 marks]

(e) (i) **METHOD 1**

consider for example $h(x) = x - 1 - \ln x$

$h(1) = 0$ and $h'(x) = 1 - \frac{1}{x}$ ***(A1)***

as $h'(x) \geq 0$ for $x \geq 1$, then $h(x) \geq 0$ for $x \geq 1$ ***R1***

as $h'(x) \leq 0$ for $0 < x \leq 1$, then $h(x) \geq 0$ for $0 < x \leq 1$ ***R1***

so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

METHOD 2

$g''(x) = -\frac{1}{x^2}$ ***A1***

$g''(x) < 0$ (concave down) for $x \in \mathbb{R}^+$ ***R1***

the graph of $y = g(x)$ is below its tangent ($y = x - 1$ at $x = 1$) ***R1***

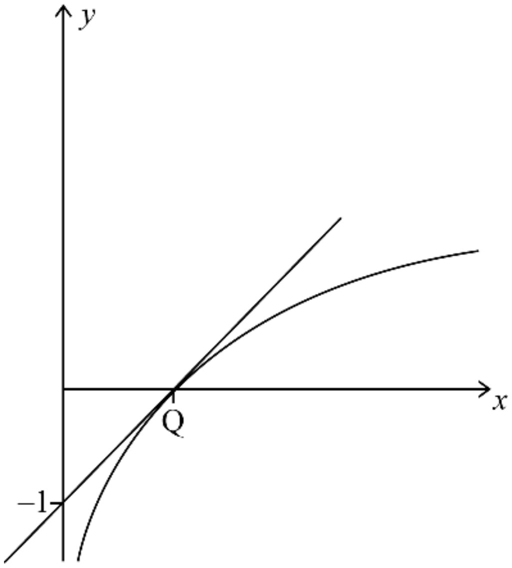
so $g(x) \leq x - 1,\ x \in \mathbb{R}^+$ ***AG***

Note: The reasoning may be supported by drawn graphical arguments.

continued …

Question 11 continued

METHOD 3



clear correct graphs of $y = x - 1$ and $\ln x$ for $x > 0$ ***A1A1***

statement to the effect that the graph of $\ln x$ is below the graph of its tangent at $x = 1$ ***R1AG***

(ii) replacing x by $\frac{1}{x}$ to obtain $\ln\left(\frac{1}{x}\right) \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***M1***

$-\ln x \le \frac{1}{x} - 1 \left(= \frac{1-x}{x}\right)$ ***(A1)***

$\ln x \ge 1 - \frac{1}{x} \left(= \frac{x-1}{x}\right)$ ***A1***

so $\frac{x-1}{x} \le g(x),\ x \in \mathbb{R}^+$ ***AG***

[6 marks]

Total [23 marks]

12. (a) (i) $\overrightarrow{\mathrm{AM}} = \frac{1}{2}\overrightarrow{\mathrm{AC}}$ ***(M1)***

$= \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

(ii) $\overrightarrow{\mathrm{BM}} = \overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{AM}}$ ***M1***

$= \boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{\mathrm{BM}} = \frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}$ ***AG***

[4 marks]

(b) (i) $\overrightarrow{\mathrm{RA}} = \frac{1}{3}\overrightarrow{\mathrm{BA}}$

$= \frac{1}{3}(\boldsymbol{a} - \boldsymbol{b})$ ***A1***

(ii) $\overrightarrow{\mathrm{RT}} = \frac{2}{3}\overrightarrow{\mathrm{RS}}$

$= \frac{2}{3}\left(\overrightarrow{\mathrm{RA}} + \overrightarrow{\mathrm{AS}}\right)$ ***(M1)***

$= \frac{2}{3}\left(\frac{1}{3}(\boldsymbol{a} - \boldsymbol{b}) + \frac{2}{3}(\boldsymbol{c} - \boldsymbol{a})\right)$ or equivalent. ***A1A1***

$= \frac{2}{9}(\boldsymbol{a} - \boldsymbol{b}) + \frac{4}{9}(\boldsymbol{c} - \boldsymbol{a})$ ***A1***

$\overrightarrow{\mathrm{RT}} = -\frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***AG***

[5 marks]

(c) $\overrightarrow{\mathrm{BT}} = \overrightarrow{\mathrm{BR}} + \overrightarrow{\mathrm{RT}}$

$= \frac{2}{3}\overrightarrow{\mathrm{BA}} + \overrightarrow{\mathrm{RT}}$ ***(M1)***

$= \frac{2}{3}\boldsymbol{a} - \frac{2}{3}\boldsymbol{b} - \frac{2}{9}\boldsymbol{a} - \frac{2}{9}\boldsymbol{b} + \frac{4}{9}\boldsymbol{c}$ ***A1***

$\overrightarrow{\mathrm{BT}} = \frac{8}{9}\left(\frac{1}{2}\boldsymbol{a} - \boldsymbol{b} + \frac{1}{2}\boldsymbol{c}\right)$ ***A1***

point B is common to $\overrightarrow{\mathrm{BT}}$ and $\overrightarrow{\mathrm{BM}}$ and $\overrightarrow{\mathrm{BT}} = \frac{8}{9}\overrightarrow{\mathrm{BM}}$ ***R1R1***

so T lies on [BM] ***AG***

[5 marks]

Total [14 marks]

13. (a) (i) METHOD 1

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = (1 + \mathrm{i}\frac{\sin\theta}{\cos\theta})^n + (1 - \mathrm{i}\frac{\sin\theta}{\cos\theta})^n$ ***M1***

$= \left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n + \left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n$ ***A1***

by de Moivre's theorem (***M1***)

$\left(\frac{\cos\theta + i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta + i\sin n\theta}{\cos^n\theta}$ ***A1***

recognition that $\cos\theta - i\sin\theta$ is the complex conjugate of $\cos\theta + i\sin\theta$ (***R1***)

use of the fact that the operation of complex conjugation commutes with the operation of raising to an integer power:

$\left(\frac{\cos\theta - i\sin\theta}{\cos\theta}\right)^n = \frac{\cos n\theta - i\sin n\theta}{\cos^n\theta}$ ***A1***

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = \frac{2\cos n\theta}{\cos^n\theta}$ ***AG***

METHOD 2

$(1 + \mathrm{i}\tan\theta)^n + (1 - \mathrm{i}\tan\theta)^n = (1 + \mathrm{i}\tan\theta)^n + (1 + i\tan(-\theta))^n$ ***(M1)***

$= \frac{(\cos\theta + i\sin\theta)^n}{\cos^n\theta} + \frac{(\cos(-\theta) + i\sin(-\theta))^n}{\cos^n\theta}$ ***M1A1***

Note: Award ***M1*** for converting to cosine and sine terms.

use of de Moivre's theorem ***(M1)***

$= \frac{1}{\cos^n\theta}(\cos n\theta + \mathrm{i}\sin n\theta + \cos(-n\theta) + \mathrm{i}\sin(-n\theta))$ ***A1***

$= \frac{2\cos n\theta}{\cos^n\theta}$ as $\cos(-n\theta) = \cos n\theta$ and $\sin(-n\theta) = -\sin n\theta$ ***R1AG***

continued ...

Question 13 continued

(ii) $\left(1+\mathrm{i}\tan\frac{3\pi}{8}\right)^4+\left(1-\mathrm{i}\tan\frac{3\pi}{8}\right)^4=\dfrac{2\cos\left(4\times\frac{3\pi}{8}\right)}{\cos^4\frac{3\pi}{8}}$ ***(A1)***

$=\dfrac{2\cos\frac{3\pi}{2}}{\cos^4\frac{3\pi}{8}}$ ***A1***

$=0$ as $\cos\frac{3\pi}{2}=0$ ***R1***

Note: The above working could involve theta and the solution of $\cos(4\theta)=0$.

so $\mathrm{i}\tan\frac{3\pi}{8}$ is a root of the equation ***AG***

(iii) either $-\mathrm{i}\tan\frac{3\pi}{8}$ or $-\mathrm{i}\tan\frac{\pi}{8}$ or $\mathrm{i}\tan\frac{\pi}{8}$ ***A1***

Note: Accept $\mathrm{i}\tan\frac{5\pi}{8}$ or $\mathrm{i}\tan\frac{7\pi}{8}$.
Accept $-\left(1+\sqrt{2}\right)\mathrm{i}$ or $\left(1-\sqrt{2}\right)\mathrm{i}$ or $\left(-1+\sqrt{2}\right)\mathrm{i}$.

[10 marks]

(b) (i) $\tan\frac{\pi}{4}=\dfrac{2\tan\frac{\pi}{8}}{1-\tan^2\frac{\pi}{8}}$ ***(M1)***

$\tan^2\frac{\pi}{8}+2\tan\frac{\pi}{8}-1=0$ ***A1***

let $t=\tan\frac{\pi}{8}$

attempting to solve $t^2+2t-1=0$ for t ***M1***

$t=-1\pm\sqrt{2}$ ***A1***

$\frac{\pi}{8}$ is a first quadrant angle and tan is positive in this quadrant, so

$\tan\frac{\pi}{8}>0$ ***R1***

$\tan\frac{\pi}{8}=\sqrt{2}-1$ ***AG***

continued …

Question 13 continued

(ii) $\cos 4x = 2\cos^2 2x - 1$ ***A1***

$= 2\left(2\cos^2 x - 1\right)^2 - 1$ ***M1***

$= 2\left(4\cos^4 x - 4\cos^2 x + 1\right) - 1$ ***A1***

$= 8\cos^4 x - 8\cos^2 x + 1$ ***AG***

Note: Accept equivalent complex number derivation.

(iii) $\int_0^{\frac{\pi}{8}} \frac{2\cos 4x}{\cos^2 x} \mathrm{d}x = 2\int_0^{\frac{\pi}{8}} \frac{8\cos^4 x - 8\cos^2 x + 1}{\cos^2 x} \mathrm{d}x$

$= 2\int_0^{\frac{\pi}{8}} 8\cos^2 x - 8 + \sec^2 x \,\mathrm{d}x$ ***M1***

Note: The ***M1*** is for an integrand involving no fractions.

use of $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ ***M1***

$= 2\int_0^{\frac{\pi}{8}} 4\cos 2x - 4 + \sec^2 x \,\mathrm{d}x$ ***A1***

$= \left[4\sin 2x - 8x + 2\tan x\right]_0^{\frac{\pi}{8}}$ ***A1***

$= 4\sqrt{2} - \pi - 2$ (or equivalent) ***A1***

[13 marks]

Total [23 marks]